A center curve under the Fréchet distance

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30.03.2020

The center curve problem

Definition

A **curve** is a continuous map $p : [0, 1] \to \mathbb{R}^m$. Let $\mathscr{C}_m$ be the set of all curves in $\mathbb{R}^m$.

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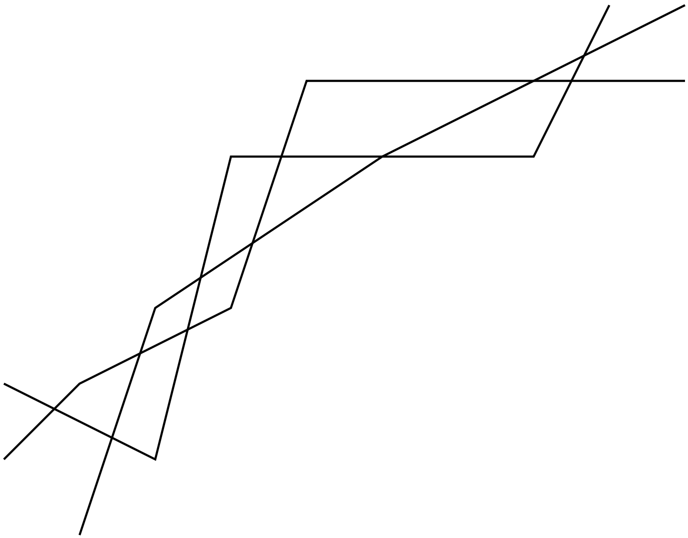
Problem

Given curves $p^1, \ldots, p^n \in \mathscr{C}_m$, $r \in \mathbb{R}$ and a distance measure $d : \mathscr{C}_m^2 \to \mathbb{R}$, find a curve $q \in \mathscr{C}_m$ such that $d(p^i, q) \leq r$ or decide that none exists.

Example Center Curve



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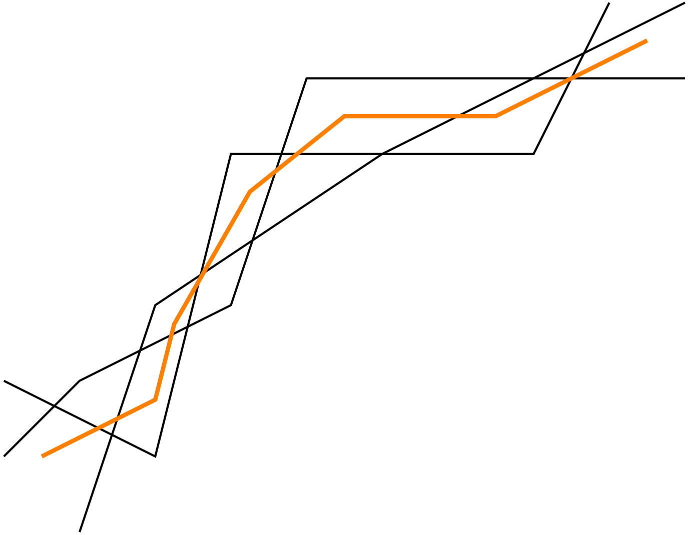

The Fréchet distance

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Define the **Fréchet distance** as the map $d_F : \mathscr{C}_m^2 \to \mathbb{R}$ with

$$d_F(p,q) := \inf_{f \in \mathscr{P}} \max_{x \in [0,1]} \|p(x) - q(f(x))\|$$

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Lemma

The Fréchet distance is symmetric and satisfies the triangle inequality. It becomes a metric by setting $p \sim q \iff d_F(p,q) = 0$.

Open questions

- The Fréchet center curve problem is *NP*-hard. Is it in *NP*?
- How can the set of solutions to the problem be represented?

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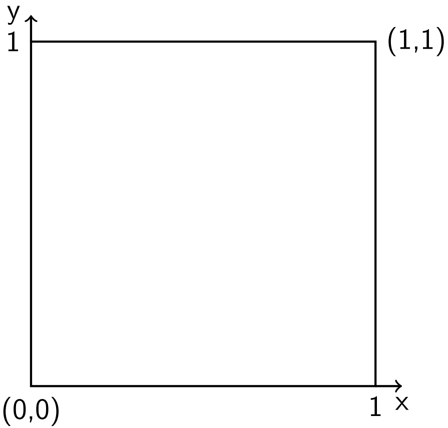

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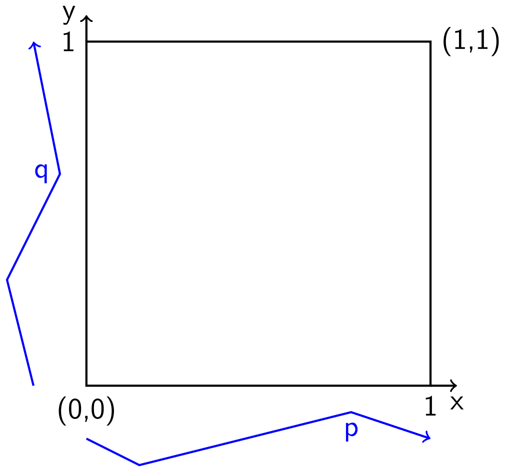

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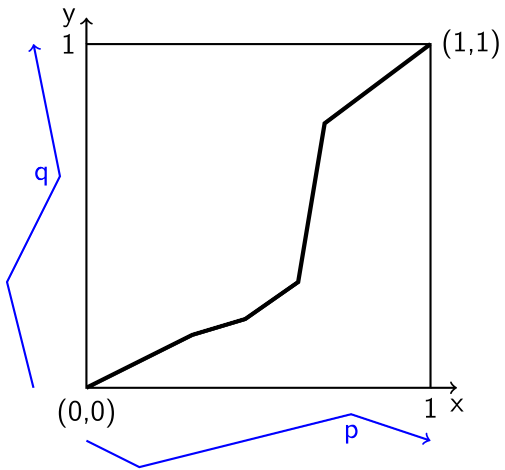

Redefining d_F with Monotone Paths

Definition

Call the image T of a curve $p_T : [0,1] \to \mathbb{R}^2$ a **monotone path** if

1. $p_T(0) = (0,0)$ and $p_T(1) = (1,1)$
2. $x \leq y \implies p_T(x)_1 \leq p_T(y)_1$ and $p_T(x)_2 \leq p_T(y)_2$.

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Denote the set of all monotone paths by $\mathscr{T}$.

Lemma

For any curves $p, q \in \mathscr{C}_m$ their Fréchet distance can be expressed as

$$d_F(p,q) = \min_{T \in \mathscr{T}} \max_{t \in T} \|p(t_1) - q(t_2)\|$$

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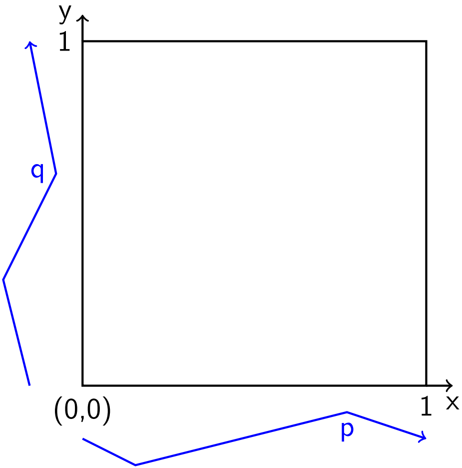

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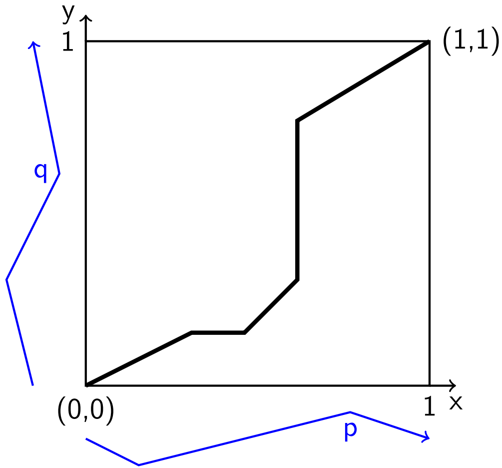

Vertex Associations

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Let p, q be piece-wise linear curves. A set $A \subset T \in \mathscr{T}$ is called a **vertex association**, if A contains a those points corresponding to vertices of either curve. Denote the set of all vertex associations by $\mathscr{A}$.

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Lemma

Let p, q be two piece-wise linear curves. Then

$$d_F = \min_{A \in \mathscr{A}} \max_{(x,y) \in A} \|p(x) - q(y)\|$$

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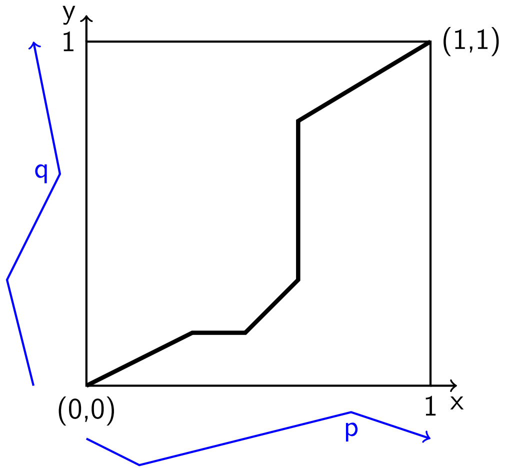

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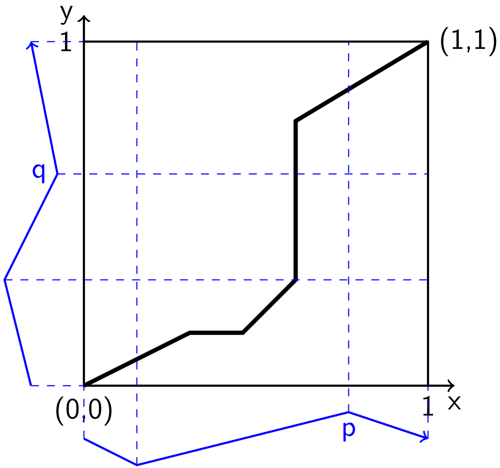

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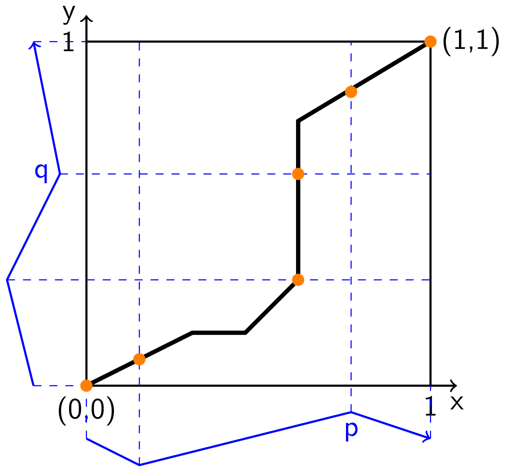

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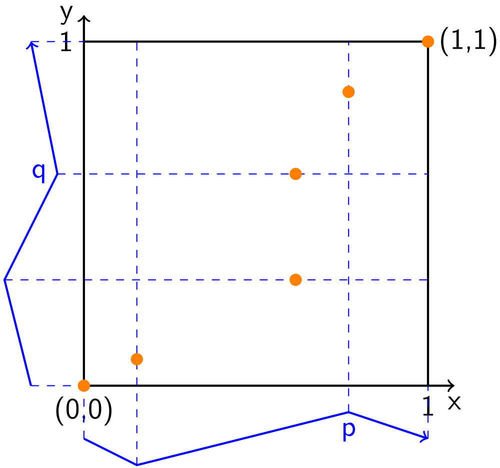

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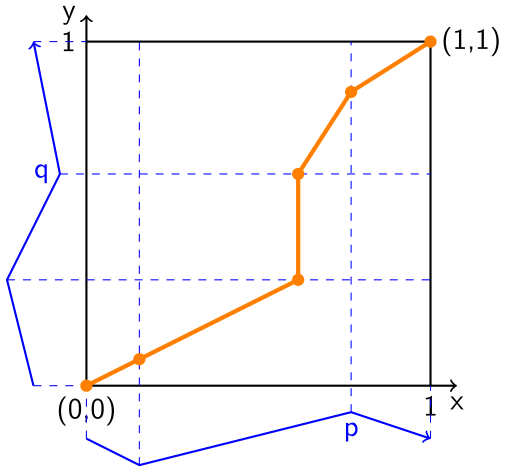

Visiting Orders

Lemma

Given piece-wise linear curves $p^1, \ldots, p^n$ each of length l and a center curve q of radius r, there is a curve q' with

- q' is piece-wise linear with $l' \leq nl$ vertices.
- $d(p^i, q') \leq r$ for all $i = 1, \ldots, n$
- $|A^i| = l'$, i.e. each vertex of p^i associates to a vertex of q'.

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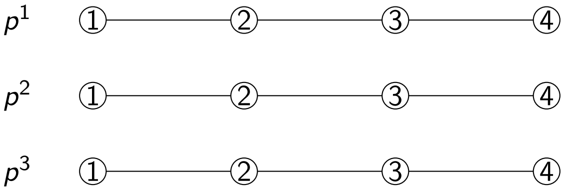
Definition

Given n piece-wise linear curves with l vertices each, let $N = n(l-2)$. Then a **visiting order** Γ is a sequence $\Gamma^0, \Gamma^1, ..., \Gamma^N \in \{1, ..., l\}^n$ where

- $\Gamma^0 = (1, ..., 1)$
- $\Gamma^k = \Gamma^{k-1} + e_{j_k}$.

Example: Visiting Order



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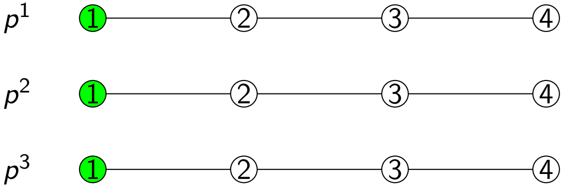

$$\Gamma^0$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

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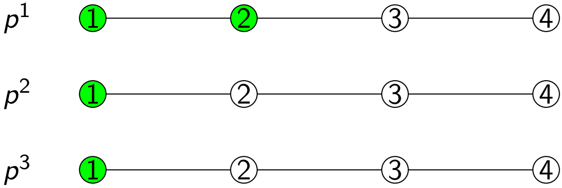

$$\Gamma^0 \quad \Gamma^1$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}2\\1\\1\end{pmatrix}$$

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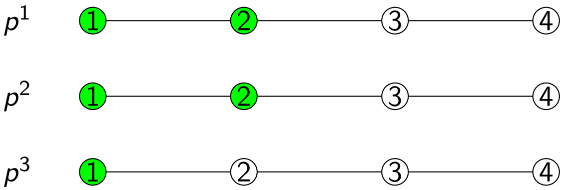

$$\Gamma^0 \qquad \Gamma^1 \qquad \Gamma^2$$
$$\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}2\\1\\1\end{pmatrix} \quad \begin{pmatrix}2\\2\\1\end{pmatrix}$$

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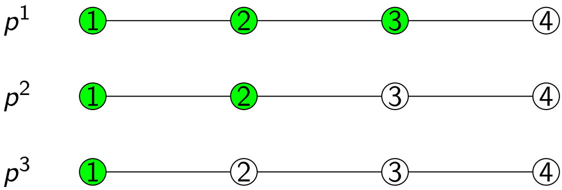

$$\Gamma^0 \quad \Gamma^1 \quad \Gamma^2 \quad \Gamma^3$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}2\\1\\1\end{pmatrix} \quad \begin{pmatrix}2\\2\\1\end{pmatrix} \quad \begin{pmatrix}3\\2\\1\end{pmatrix}$$

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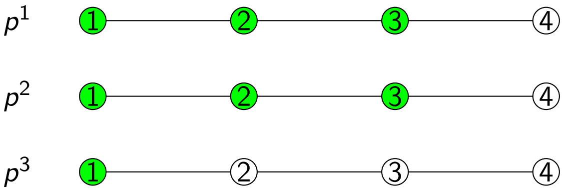

$$\Gamma^0 = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad \Gamma^1 = \begin{pmatrix}2\\1\\1\end{pmatrix} \quad \Gamma^2 = \begin{pmatrix}2\\2\\1\end{pmatrix} \quad \Gamma^3 = \begin{pmatrix}3\\2\\1\end{pmatrix} \quad \Gamma^4 = \begin{pmatrix}3\\3\\1\end{pmatrix}$$

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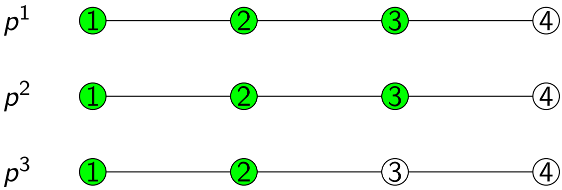

Γ^0	Γ^1	Γ^2	Γ^3	Γ^4	Γ^5
$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}2\\1\\1\end{pmatrix}$	$\begin{pmatrix}2\\2\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\\1\end{pmatrix}$	$\begin{pmatrix}3\\3\\1\end{pmatrix}$	$\begin{pmatrix}3\\3\\2\end{pmatrix}$

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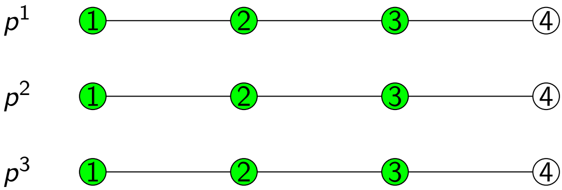

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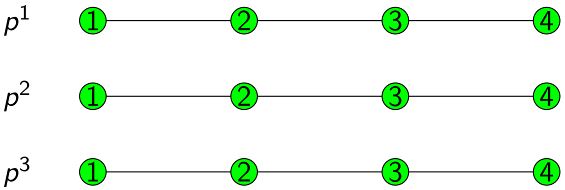

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- $\lambda^i_k = 0$ if $\Gamma^k_i - \Gamma^{k-1}_i = 1$ (say $i = j_k$).

The Inequality System

The set of solutions to the center curve problem is represented as

$$Q_\Gamma(p,r) = \left\{ \begin{array}{lrcr} (q,\lambda) \in \mathbb{R}^{N\times m + n\times N} : & & & \\ & \|q_k - s_k^i\| & \leq & r \\ (1-\lambda_k^i)p^i_{\Gamma_i^k} - \lambda_k^i p^i_{\Gamma_i^k+1} & & = & s_k^i \\ & \|q_0 - p_1^i\| & \leq & r \\ & \|q_{N+1} - p_l^i\| & \leq & r \\ & \lambda_k^{j_k} & = & 0 \\ \forall i \neq j_k & \lambda_{k-1}^i & \leq & \lambda_k^i \\ & 0 \leq \lambda_k^i & \leq & 1 \end{array} \right\}$$

where $k = 1, ..., N$ and $i = 1, ..., n$ in each line.

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$Q_\Gamma(p,r)$ is convex. $\implies$ Use or adapt convex optimization techniques!

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- If the λ_k^i are known, this reduces to solving $N + 2$ center problems of points.
- If the q_k are known, this reduces to computing n vertex associations for a given visiting order. This can be done efficiently by a greedy algorithm.
- If the constraints $\lambda_{k-1}^i \leq \lambda_k^i$ are dropped, all q_k are independent from each other.
  $\implies$ Each is the solution to a center problem of points and lines.

The Ellipsoid Method

Separation

Each inequality constraining $Q(p, r)$ can be made into a polynomial time separation oracle.
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Proof.

Given a vector y the constraint $y^T y \leq r^2$ can be checked efficiently.
If $\|y\| > r$, for any $x \in Q_\Gamma(p, r)$, by Cauchy-Schwarz:

$$y^T x \leq \|y\|\|x\| \leq \|y\| r < \|y\|^2 = y^T y$$

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to get a point $x \in Q_\Gamma(p, r)$ or decide that $vol(Q_\Gamma(p, r)) < C$ for any $C > 0$.

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- If $x \in Q_\Gamma^\epsilon(p,r)$ is found, then $x \in Q_\Gamma(p,r+\epsilon)$ with small changes.

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- Let $Q^{\epsilon}_{\Gamma}(p, r)$ result from $Q_{\Gamma}(p, r)$ by increasing all right hand sides by ϵ.
- If $Q_{\Gamma}(p, r) \neq \emptyset$, then $vol(Q^{\epsilon}_{\Gamma}(p, r)) \geq C(m, \epsilon)$.
- If $x \in Q^{\epsilon}_{\Gamma}(p, r)$ is found, then $x \in Q_{\Gamma}(p, r + \epsilon)$ with small changes.

Theorem

For inputs p, r, fixed visiting order σ and $\epsilon > 0$ there is a polynomial time algorithm that

- gives a center curve with radius $\leq r + \epsilon$ or
- decides there is no center curve of radius r.

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Lemma

Each linear closed constraint of a closed convex set P is attained with equality by some point $x \in P$ unless it is redundant.

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Each linear closed constraint of a closed convex set P is attained with equality by some point $x \in P$ unless it is redundant.

Proposition

By guessing which are attained as equalities, all linear inequalities constraining $Q_\Gamma(p, r)$ can be transformed or removed.

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Each linear closed constraint of a closed convex set P is attained with equality by some point $x \in P$ unless it is redundant.

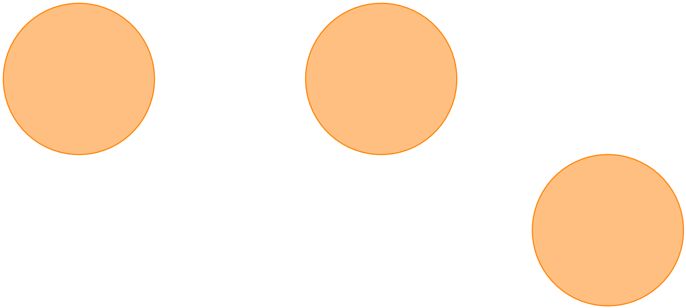
Proposition

By guessing which are attained as equalities, all linear inequalities constraining $Q_\Gamma(p, r)$ can be transformed or removed.

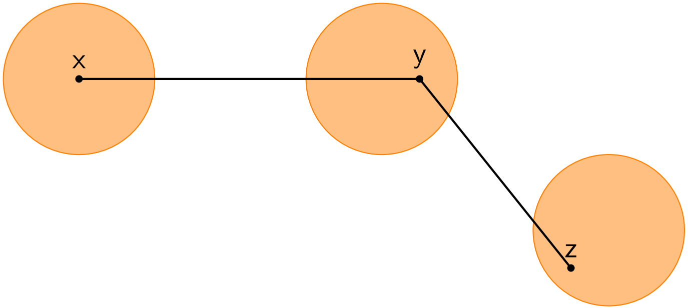
Question

Can a similar thing be done for nonlinear inequalities?

Assumptions on nonlinear constraints



Assumptions on nonlinear constraints



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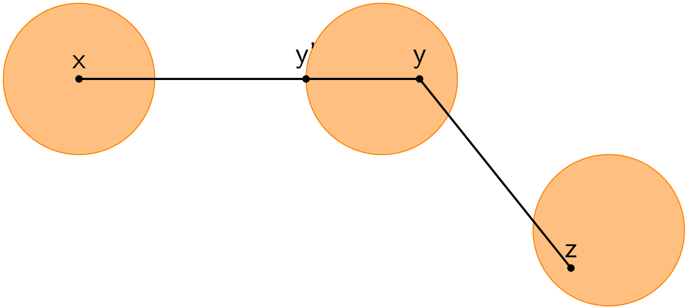

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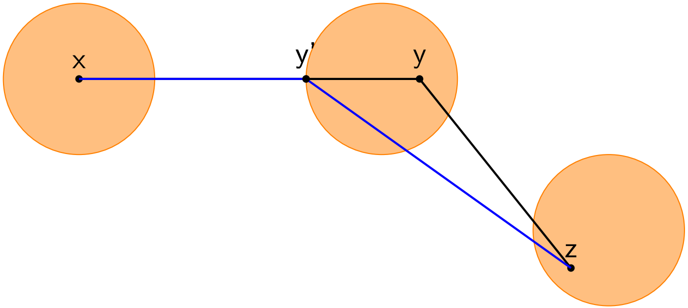

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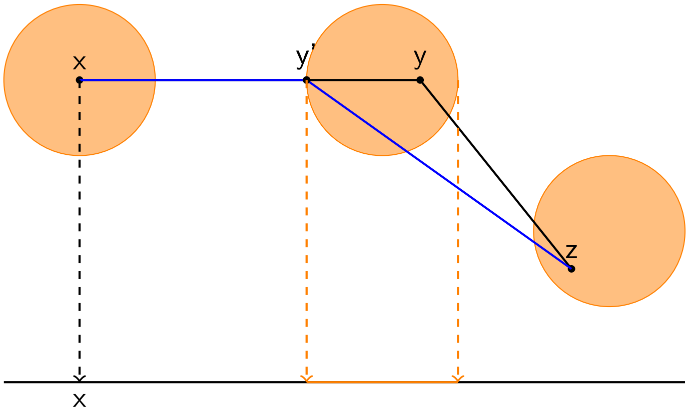

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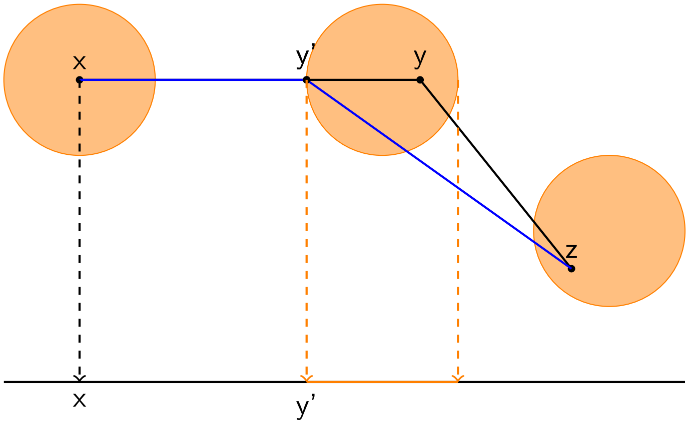

Observations on the line

Lemma

A center curve q can be chosen such that for all $k = 1, \ldots N$, one of the conditions $q_k = q_{k-1}$ or $\| q_k - p^{\sigma_k}_{\Gamma^k_{\sigma_k}} \| = r$ is satisfied.

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Fact

For a point $y \in \mathbb{R}$, there are exactly two points $y_1 < y_2 \in \mathbb{R}$ with $|y_i - y| = r$ for any $r > 0$.

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Idea

Build the center curve iteratively by either staying at the current vertex or moving to the closer one of the two candidates.

A center curve Algorithm on the Line

Theorem

Given curves $p^1, \ldots, p^n$ in $\mathbb{R}$ and $r \geq 0$, a curve q can be computed in time $O(N)$ such that q is a center curve with radius r if one exists.

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Lemma

Whether $d_F(p^i, q) \leq r$ can be checked in polynomial time.

Counting Visiting Orders

Lemma

Given n curves with l vertices each, the number $|\mathscr{V}|$ of possible visiting orders is $|\mathscr{V}| = \frac{N!}{(l-2)!^n}$.

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- An algorithm trying all visiting orders can have exponential, but no polynomial runtime.
- Is there a way to not test every single visiting order?

n-wise Fréchet distance

Definition

Given curves $p^1, \ldots, p^n$ their n-wise Fréchet distance is defined as

$$d_F(p^1, \ldots, p^n) = \inf_{f_1, \ldots, f_n \in \mathscr{P}} \max_{1 \le i,j \le n} \max_{x \in [0,1]} \| p^i(f_i(x)) - p^j(f_j(x)) \|$$

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- Strong connection to center curve problem.
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- Can be solved by guessing v.o. and checking a center curve candidate.
- The n-wise distance is at most twice the center curve radius.

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- A vertex association can be computed iteratively for any visiting order, even if it is a partial order.

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- A vertex association can be computed iteratively for any visiting order, even if it is a partial order.
- The n-wise distance on a partial order is a lower bound to the distance on any extension of that order.
- $\implies$ The n-wise distance can be decided by Branch-and-Bound on visiting orders.
- Using $2r$ as a bound allows to decide if a visiting order could attain center curve radius r.

Thank you for listening!