Thomas Kesselheim
Anna Heuser

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Due date: May 9, 2023 at 10:00 p.m.

Algorithmic Game Theory

Summer Term 2023

Exercise Set 5

If you want to hand in your solutions for this problem set, please send them via email to anna.heuser@uni-bonn.de *by Tuesday evening – make sure to send a pdf-file which contains your name and your email address. Of course, submitting solutions in groups is also possible.*

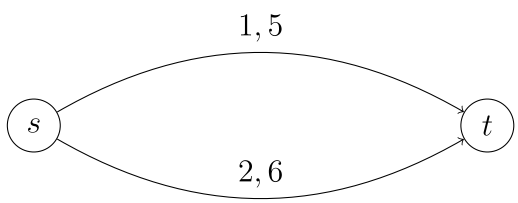
If you would like to present one of the solutions in class, please also send an email to anna.heuser@uni-bonn.de *containing the* ***task*** *which you would like to present and in* ***which of the tutorials*** *you would like to do so. Deadline for the email is Tuesday, 10:00 pm. Please note that the tasks will be alloecated via a first-come-first-served procedure, so sending this email earlier than Tuesday evening is highly recommended.*

Exercise 1: (1+3+2 Points)

Referring to the price of anarchy from Lecture 8 we can introduce a more optimistic point of view called the *price of stability*. For an equilibrium concept Eq, it is defined as

$$PoS_{\mathsf{Eq}} = \frac{\min_{p \in \mathsf{Eq}} SC(p)}{\min_{s \in S} SC(s)} \quad .$$

Consider the following symmetric network congestion game with two players:



(a) What is the Price of Anarchy and the Price of Stability of pure Nash equilibria?

(b) What is the Price of Anarchy and the Price of Stability of mixed Nash equilibria?

(c) What is the best upper bound for the Price of Anarchy that can be shown via smoothness?

Exercise 2: (6 Points)

Consider a (λ, μ)-smooth game with N players and let $s^{(1)}, \dots, s^{(T)}$ be a sequence of states such that the external regret of every player is at most $R^{(T)}$. Moreover, let s^* denote a state that minimizes the social cost. We want to upper bound the average social cost of the sequence of states. To this end, show the following bound

$$\frac{1}{T} \sum_{t=1}^{T} SC(s^{(t)}) \leq \frac{N \cdot R^{(T)}}{(1-\mu)T} + \frac{\lambda}{1-\mu} SC(s^*)\, .$$

Exercise 3: (2+2 Points)

We call s an ϵ-approximation to a pure Nash equilibrium if $c_i(s) \leq (1+\epsilon)c_i(s_i', s_{-i})$ for all i and s_i'.

(a) Consider a (λ, μ)-smooth cost-minimization game and let and $0 < \epsilon < \frac{1}{\mu} - 1$. Prove that the PoA of ϵ-approximations to pure Nash equilibria is at most $\frac{(1+\epsilon)\lambda}{1-(1+\epsilon)\mu}$.

(b) Can you state a similar result for more general equilibrium concepts?

Exercise 4: (2 Points)

A *fair cost-sharing game* is a congestion game such that for all resources $r \in \mathcal{R}$ the delay function can be modeled as $d_r(x) = c_r/x$ for a constant c_r.

Show that fair cost sharing games with n players are $(n, 0)$-smooth.

For presentation consider Exercise 3 and Exercise 4 to be one exercise, hence they need to be presented together!