Thomas Kesselheim
Anna Heuser

May 8, 2023
Due date: May 23, 2023 at 10:00 p.m.

Algorithmic Game Theory

Summer Term 2023

Exercise Set 6

If you want to hand in your solutions for this problem set, please send them via email to anna.heuser@uni-bonn.de *by Tuesday evening – make sure to send a pdf-file which contains your name and your email address. Of course, submitting solutions in groups is also possible.*

If you would like to present one of the solutions in class, please also send an email to anna.heuser@uni-bonn.de *containing the* ***task*** *which you would like to present and in* ***which of the tutorials*** *you would like to do so. Deadline for the email is Tuesday, 10:00 pm. Please note that the tasks will be alloecated via a first-come-first-served procedure, so sending this email earlier than Tuesday evening is highly recommended.*

Exercise 1: (2+3 Points)
Consider a second-price auction with a fixed value profile $(v_i)_{i \in N}$. Since the value profile is fixed, we get a normal-form utility-maximization game.

(a) Show that there exists a pure Nash equilibrium in the defined game.

(b) Now, consider a game in which only two players participate and $v_1 \gg v_2$ holds. Prove that even in this setting there exists a pure Nash equilibrium such that bidder 2 wins.

Exercise 2: (4+4 Points)
We consider an auction of k identical items. Each bidder can acquire at most one of the items. If bidder i gets one of the items, she has a value of v_i. Otherwise, that is, if she does not get an item, she has a value of 0.

(a) State a generalization of the second-price auction and prove that it is truthful (the second-price auction covers the case of $k = 1$). Follow steps in the spirit of Lecture 10.

(b) Now, consider a mechanism which sequentially performs k second-price auctions. That is, initially each bidder reports one bid. Then, in each auction, one item is sold among the remaining players using their initial bids. Show that truthful bidding does not necessarily lead to a pure Nash equilibrium even in the special case of three players and $k = 2$.

Exercise 3: (3+1 Points)
A billionaire considers selling tours to the moon. The cost of building a rocket is C. Let $N = \{1, \ldots, n\}$ be the set of people who initially have declared an interest in the trip. The billionaire wishes to design a mechanism that will recover his cost but does not have information about the private valuation the bidders have for joining the trip. Therefore, he runs the following auction given as pseudocode:

- All bidders $i \in N$ simultaneously submit their bids $b_i \geq 0$.
- $S \leftarrow N$
- While $S \neq \emptyset$ do
  - $S' \leftarrow \{i \in S \mid b_i \geq \frac{C}{|S|}\}$
  - If $S' = S$, then allocate a seat for each $i \in S$ and no seat for each $i \in N \setminus S$. All bidders $i \in S$ have to pay $\frac{C}{|S|}$. The rest of the bidders $i \in N \setminus S$ has to pay nothing. Return.
  - Otherwise, $S \leftarrow S'$
- Do not allocate any seat and charge no payments at all. Return.

(a) Show that the described mechanism is truthful.

(b) Show that if the bidders are truthful, the auction finds the largest set of bidders that can share the target cost C equally, if there is one.

Exercise 4: (5 Points)
Recall the *Greedy-by-Value* and *Greedy-by-Sqrt-Value-Density* algorithms for single-minded CAs of lecture 12. Let us analyse another greedy algorithm that looks as follows.

Greedy-by-Value-Density

- Re-order the bids such that $\frac{b_1^*}{|S_1^*|} \geq \frac{b_2^*}{|S_2^*|} \geq \cdots \geq \frac{b_n^*}{|S_n^*|}$.
- Initialize the set of winning bidders to $W = \emptyset$.
- For $i = 1$ to n do: If $S_i^* \cap \bigcup_{j \in W} S_j^* = \emptyset$, then $W = W \cup \{i\}$.

Let $d = \max_{i \in \mathcal{N}} |S_i^*|$. Show that the given algorithm yields a d-approximation.

Exercise 5: (5 Points)
Consider a *Knapsack Auction* which is defined the following way. Each bidder i has a publicly known weight w_i and a private value v_i. A feasible outcome is any set S of bidders such that $\sum_{i \in S} w_i \leq W$ holds for a fixed bound W. Furthermore, we assume that $0 \leq w_i \leq W$ for all bidder i.
The following algorithm yields a 2-approximation:

- Sort and renumber the bidders such that $\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \ldots \frac{b_n}{w_n}$. Let k be the largest integer such that $\sum_{i=1}^{k} w_i \leq W$ and set $S_1 = \{1, \ldots, k\}$.
- Let i^* be the bidder with the maximum bid b_i among all bidders and set $S_2 = \{i^*\}$.
- Return the better solution of S_1 and S_2.

Show that the given algorithm is monotone and state a truthful mechanism with the aid of Myerson's Lemma.