Thomas Kesselheim
Alexander Braun

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Due date: December 22, 2021 at 10:00 a.m.

Algorithmic Game Theory

Winter Term 2021/22

Exercise Set 10

Exercise 1: (6 Points)

We work in the setup of combinatorial auctions with m (possibly heterogeneous) items M. Bidders report bids for items and afterwards each item is sold in a separate second-price auction (item bidding). Prove the following theorem.

Theorem. Consider a pure Nash equilibrium b of item bidding with second-price payments and unit-demand bidders. Let $X_1, \ldots, X_n$ be the resulting allocation. If for all bidders i we have $\sum_{j \in X_i} b_{i,j} \leq v_i(X_i)$ (weak no-overbidding), then $\sum_{i \in N} v_i(X_i) \geq \frac{1}{2} OPT(v)$.

Hint: Make use of the following deviation bids: Consider the welfare-maximizing allocation on v. Let j_i be the item that is assigned to bidder i in this allocation. If i does not get any item, set j_i to $\perp$. Set $b^*_{i,j} = v_{i,j}$ if $j = j_i$ and 0 otherwise. Now, derive a proof in the spirit of Theorem 17.2.

Exercise 2: (4 Points)

Consider m items and n bidders. We define a generalization of Walrasian equilibria: Let $S = (S_1, \ldots, S_n)$ be an allocation of items to bidders and $q \in \mathbb{R}^m_{\geq 0}$ be a price vector. We call the pair (q, S) an *ϵ-approximate Walrasian equilibrium* if unallocated items have price 0, every bidder i has non-negative utility $v_i(S_i) - \sum_{j \in S_i} q_j \geq 0$, and every bidder receives items within ϵ of its favorite bundle, i.e., $v_i(S_i) - \sum_{j \in S_i} q_j \geq v_i(S'_i) - \sum_{j \in S'_i} q_j - \epsilon$ for every bundle S'_i.

Prove an approximate version of the First Welfare Theorem: If (q, S) is an ϵ-approximate Walrasian equilibrium, then the social welfare of an optimal allocation S^* cannot surpass the one of S by more than $\min\{m, n\} \cdot \epsilon$.

Exercise 3: (4 Points)

Recall the valuation functions of single-minded bidders from Definition 12.2. Let the maximum bundle size be defined by $d = \max_{i \in \mathcal{N}} |S^*_i|$. Show that in the case of single-minded bidders with maximum bundle size d, item bidding with first price payments is $(\frac{1}{2}, 2d)$-smooth.

Hint: In order to define deviation bids $b^*_{i,j}$, consider a welfare-maximization allocation on v. If bidder i does not get his bundle in the optimal allocation, then define $b^*_{i,j} = 0$ for all items $j \in M$. Otherwise, define $b^*_{i,j} = \frac{v_i}{2d}$ for all $j \in S^*_i$ and $b^*_{i,j} = 0$ if $j \notin S^*_i$. That is, each winner in the optimal allocation equally divides the value for his bundle among all items of the bundle and bids half of it.