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October 23, 2023
Due: October 30, 2023 at 10:00 p.m.

Algorithms and Uncertainty

Winter Term 2023/24

Exercise Set 3

If you want to hand in your solutions for this problem set, please send them via email to anna.heuser@uni-bonn.de *by Monday evening – make sure to send a pdf-file which contains your name and your email address. Of course, submitting solutions in groups is also possible.*

If you would like to present one of the solutions in class, please also send an email to anna.heuser@uni-bonn.de *containing the* ***task*** *which you would like to present and in* ***which of the tutorials*** *you would like to do so. Deadline for the email is Monday, 10:00 pm. Please note that the tasks will be allocated via a first-come-first-served procedure, so sending this email earlier than Monday evening is highly recommended.*

Exercise 1: (5 Points)

Consider the following randomized rounding for fractional Ski Rental. In step t, flip an independent biased coin: With probability $\frac{x_{\text{buy}}^{(t)} - x_{\text{buy}}^{(t-1)}}{1 - x_{\text{buy}}^{(t-1)}}$ buy the skis, otherwise rent them. Show that if the underlying fractional algorithm is α-competitive so is the randomized integral algorithm.

Exercise 2: (4 Points)

We would like to show that optimal online algorithms that know the input distribution in advance are always deterministic.

To this end, we use the notation of Lecture 6. We have a finite class of deterministic algorithms $\mathcal{A}$ and a finite class of instances $\mathcal{X}$. Let X be a random variable with values in $\mathcal{X}$ and A be an *independent* random variable with values in $\mathcal{A}$. Show that there is a $a \in \mathcal{A}$ such that $\mathbf{E}[c(a, X)] \leq \mathbf{E}[c(A, X)]$.

Hint: Use the same techniques as in the proof of Yao's principle.

Exercise 3: (1+5 Points)

Consider the following online problem: We would like to buy a house. Each time we view a house, we get to know how good it is. Immediately after, we have to decide whether to buy the current house or not. If we do not buy it now, there will be no other opportunity later. Formally, this means that a sequence of values is revealed one after the other. Note that there are no assumptions concerning the order of the sequence. In particular, the order could be chosen by an adversary. The algorithm can always decide if it stops the sequence at its current value or if it waits. We say that an algorithm wins if it stops the sequence at its highest value.

(a) Assume that the length of the sequence n is known. Give a randomized online algorithm that wins with probability $\frac{1}{n}$.

Hint: Trivial.

(b) Use Yao's principle to show that no algorithm wins with higher probability.

Hint: Use the profit-maximization variant of Yao's principle. The profit is 1 if the algorithm wins, 0 otherwise. The offline optimum is clearly always 1. Construct a probability distribution over sequences of the form $1, 2, \ldots, t, 0, \ldots, 0$.